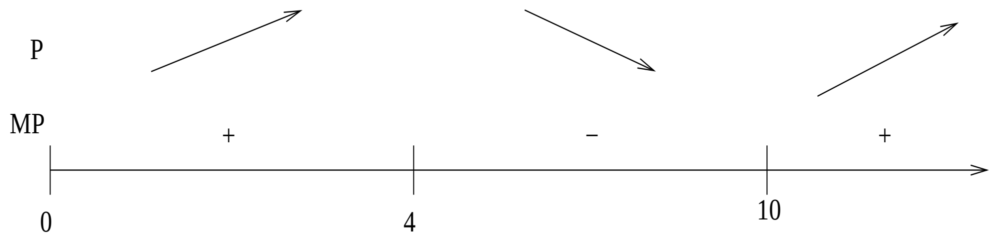
Math 211

Review Problems

1. A company's marginal profit is $MP(x) = 3(x^2 - 14x + 40)$ dollars/unit if they make x units of their product $(0 \le x \le 15)$

   (a) Find the interval in which p(x) is an increasing and a decreasing function of x, and display them on the number line below.

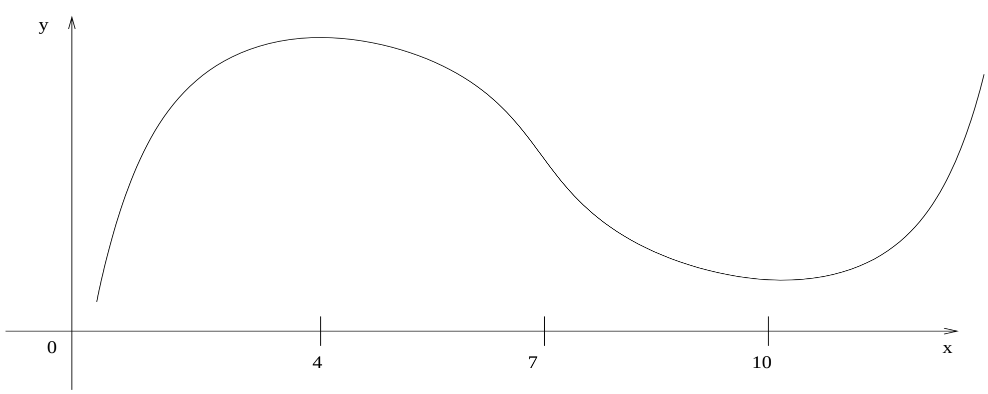
   (**Solution**)$MP(x) = P'(x) = 3(x^2 - 14x + 40) = 3(x-10)(x-4)$. Set equal to zero to get $x = 4$ and $x = 10$



   (b) Find the intervals in which the graph of $y = P(x)$ is concave upward and concave downward, and display these and any point of inflection on the line below.

   (**Solution**)$P''(x) = 3(2x - 14)$. Set equal to zero to find $x = 7$.

   (c) Sketch the graph of $y = P(x)$ for $x \ge 0$, given the above data and the fact that $P(x) > 0$ whenever $x \ge 0$.



   (d) The company's profit from selling 3 units is \$198. What is their approximate profit if they sell 3.1 units? [Do Not use a calculater]
   (**Solution**)By Linear approximation,

$$P(3.1) \approx P(3) + P'(3.0)(3.1 - 3) = 198 + 21(0.1) = 200.1$$

2. A producer of fashion watches determines that the monthly revenue R for one of its models (in \$) is related to the selling price p (in \$) by the equation

$$R = 1000p - 4p^{7/4} - 252$$

(a) Find the marginal revenue at price level $p_0 = \$81$. (Include the correct units in your answer.)
(**Solution**)$MR(p) = 1000 - 7(p)^{3/4}$. $MP(81) = 1000 - 7(81)^{3/4} = 811(\$/\$)$, or no unit.

(b) Use linear approximation to estimate the revenue resulting from an increase in price from $p_0 = \$81$ to $p = \$83$.
(**Solution**)

$$R(83) \approx R(81) + MR(81)(83 - 81) = 73622$$

(c) The company would like its monthly revenue from this model of watch to be \$75,000. Approximately how much should they charge per watch, according to your work in part **b**)?
(**Solution**)Solve $75000 \approx R(81) + MR(81)(p - 81)$ or $75000 \approx 72000 + 811(p - 81)$. Then $p \approx 84.6$(dollar).

(d) Explain briefly why it would be more difficult to answer part **c**) without linear approximation.
(**Solution**)Without linear approximation, we need to solve

$$75000 = 1000p - 4p^{7/4} - 252$$

which is very hard to solve.

3. The demand x (in gallons) for a very expensive fuel offered for sale in a certain market is related to the price p per gallon (in \$) by the formula $x^2 + 2xp + 3p^2 = 1800$.

(a) At what rate is x changing when p is increasing at the rate of \$3 per day, p=10, and $x = 30$? (State the correct units in your answer.)
(**Solution**)By implicit differentiation,

$$2x\frac{dx}{dt} + 2\frac{dx}{dt}p + 2x\frac{dp}{dt} + 6p\frac{dp}{dt} = 0$$

Solve for $\frac{dx}{dt}$ to get

$$\frac{dx}{dt} = \frac{-(2x + 6p)\frac{dp}{dt}}{2x + 2p} = \frac{-(2(30) + 6(10))(3)}{2(30) + 2(10)} = -4.5$$

(b) At what rate is x changing per dollar increase in p when $p = 10$ and $x = 30$?
(**Solution**)By implicit differentiation with respect to p

$$2x\frac{dx}{dp} + 2\frac{dx}{dp}p + 2x + 6p = 0$$

So,

$$\frac{dx}{dp} = \frac{-(2x + 6p)}{2x + 2p} = -1.5$$

(c) Based on your answer to part **b**), what is the approximate demand for fuel if the price is \$10.50 per gallon? State the appropriate units in your answer, and give the answer to 2 decimal places.
(**Solution**)

$$x(10.50) \approx x(10) + \frac{dx}{dp}(10)(10.50 - 10) = 30 + (-1.93)(0.50) = 29.0$$

4. A manufacturer of men's shoes experiences a total cost of $C(x) = 200+15x+(x^3/4)$ dollars in producing x pairs of shoes per day ($4 \le x \le 12$). If the manufacturer receives \$90 for each pair of shoes produced, what production levels yield maximum and minimum profit?

   (Solve this problem by using calculus, *not* by trial and error.)

   (**Solution**)$R(x) = 90x$ and

$$P(x) = 90x - 200 - 15x - \frac{x^3}{4} = 75x - 200 - \frac{x^3}{4}$$

   Then $P'(x) = 75 - \frac{3}{4}x^2$ and set equal to zero to find $x^2 = 100$ and $x = 10$, $x = -10$. Since $x >= 4$, $x = 10$ is the only critical point and $P(10) = 300$

   At the two end points, $P(4) = 84$ and $P(12) = 268$. So, maximum profit is 300 dolllars and minimum prifit is 84 dollars.

5. Find $\frac{dy}{dx}$. DO NOT SIMPLIFY your answer.

   (a) $y = x^2 e^{[x^3-2x]^3}$
   (**Solution**)

$$\frac{dy}{dx} = 2xe^{[x^3-2x]^3} + x^2 e^{[x^3-2x]^3}(3[x^3-2x]^2(3x^2-2))$$

   (b) $y = [\ln(x^2+3)]e^{(x^2+5x+1)^6}$

$$\frac{dy}{dx} = \frac{2x}{x^2+3}e^{(x^2+5x+1)^6} + \ln(x^2+3)e^{(x^2+5x+1)^6}(6(x^2+5x+1)^5(2x+5))$$

   (c) $y = (2x+3)^e + e^{3x+4} + (4x+5)^{6x+7}$
   (**Solution**)Let $u = (2x+3)^e + e^{3x+4}$ and $v = (4x+5)^{6x+7}$. We find $\frac{du}{dx}$ and $\frac{dv}{dx}$ first, then add two to get $\frac{dy}{dx}$

$$\frac{du}{dx} = e(2x+3)^{e-1}(2) + e^{3x+4}(3)$$

   To find $\frac{dv}{dx}$ we do logarithmic differentiation.

$$\ln v = \ln(4x+5)^{6x+7} = (6x+7)\ln(4x+5)$$

   Take the (implicit) derivative with respect to x to get

$$\frac{1}{v}\frac{dv}{dx} = 6\ln(4x+5) + (6x+7)\frac{4}{4x+5}$$

   and

$$\frac{dv}{dx} = (4x+5)^{6x+7}\left(6\ln(4x+5) + (6x+7)\frac{4}{4x+5}\right)$$

   Finally, add $\frac{du}{dx}$ and $\frac{dv}{dx}$ above to get

$$\frac{dy}{dx} = e(2x+3)^{e-1}(2) + e^{3x+4}(3) + (4x+5)^{6x+7}\left(6\ln(4x+5) + (6x+7)\frac{4}{4x+5}\right)$$

6. A finance student, R.V. Winkle, decided to invest \$ 100 in an account and then take a 100 year nap, after which there would be \$ 100,000 in the account. Assuming that the interest rate does not change during that nap, what NAR (nomial annual rate) of interest, compounded continuously, would Winkle want?

   (**Solution**)Let $P(t)$ be the amount after t years and $P_0 = P(0)$. Then $P(100) = 100000$ and $P_0 = 100$. Let r be teh NAR. Since we have

$$P(t) = P_0 e^{rt}, \qquad 100000 = 100e^{100r}, \qquad e^{100r} = 1000$$

   Take ln to get $100r = \ln 1000$ or $r = \ln 1000/100 = 0.069$ or 6.9%.

7. When a foreign substance is introduced into the body,the body's defense mechanisms move to break down the substance and excrete it. The rate of excretion is usually proportional to the concentration in the body,and the half-life of the resulting exponential decay is referred to as the *biological half-life* of the substance.
If,after 12 hours,15 % of a massive dosage of a substance has been excreted by the body,what is the biological half-life of the substance?

(**Solution**)Let $A(t)$ be the amount of the substance after time t hours and $A_0 = A(0)$ and r be the rate of excretion. Since the rate of excretion is proportional to the concentration in the body, we have the relation

$$A'(t) = rA(t), \text{ or } A(t) = A_0e^{rt}$$

We know that when $t = 12$, $A = 0.85A_0$. So,

$$0.85A_0 = A(12) = A_0e^{12r}, \qquad 0.85 = e^{12r}$$

By taking ln both sides, we get

$$r = \frac{\ln 0.85}{12} = -0.0135$$

Now, we need to find the half-life, i.e. to find the time t at which $A(t) = 0.5A_0$.

$$0.5A_0 = A_0e^{rt}, \qquad e^{rt} = 0.5$$

Where r is the value we found above. By taking ln both sides, we get $t = \frac{\ln 0.5}{r} = \frac{12 \ln 0.5}{\ln 0.85} = 51.18$ hours. So, the half-life is about 51.18 hours.

8. $x^2 + y^3 - 2x^3y^2 = -596$.

   (a) Find $\frac{dy}{dx}$ when $x = 2$ and $y = 10$.
   (**Solution**)By implicit differentiation with respect to x

$$2x + 3y^2\frac{dy}{dx} - 6x^3y^2 - 4x^3y\frac{dy}{dx} = 0$$

   By solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2x + 6x^2y^2}{3y^2 - 4x^3y} = -119.8$$

   (b) Use calculus to find the approximate value of y when $x = 1.9$.
   (**Solution**)Use the linear approximation to get,

$$y(1.9) \approx y(2) + y'(2)(1.9 - 2) = 10 + (-119.8) * (-0.1) = 21.98$$

9. Your chunky-style ketchup factory has just purchased a giant funnel in the form of a stainless steel cone,to store ketchup prior to dispensing it(from the tip).The volume of ketchup in this cone is $V = 3h^3$ cubic feet,where h is the height of the ketchup above the tip the cone,in feet.
The cone is being filled at a steady 40 cubic feet per minute,and is now filled to a height of 10 feet.At what rate is the gooey mass now rising in the cone?(State the appropriate units)

(**Solution**)We know that $V = 3h^3$ and $\frac{dV}{dt} = 40$. We want to find $\frac{dh}{dt}$ when $h = 10$. So, by implicit defferentiation on $V = 3h^3$ with respect to t, we get

$$\frac{dV}{dt} = 9h^2\frac{dh}{dt}, \qquad \frac{dh}{dt} = \frac{1}{9h^2}\frac{dV}{dt}$$

We know that $\frac{dV}{dt} = 40$ and we want to find this when $h = 10$. So, $\frac{dV}{dt} = \frac{1}{9(10^2)}(40) = 0.0444$(feet/minute)

10. The demand Q for a product is related to its price per ton by $Q = 90 - (0.4)p$. At what price(s) is the demand neither elastic nor inelastic?

(**Solution**)We have

$$E(p) = -\frac{pQ'(p)}{Q(p)} = \frac{0.4p}{90 - 0.4p}$$

The price where the damand neither elastic nor inelastic is when $E(p) = 1$ or $\frac{0.4p}{90-0.4p} = 1$. Solving this, we have $p = \frac{90}{0.8} = 112.5$

(NOTE) It is a good exercise to find where it is elastic or inelastic.

11. A manufacturer's marginal cost function is

$$MC(x) = 100 + 4x$$

and the total cost at production level $x = 100$ is \$35,000.
Find the total cost function.

(**Solution**)Find the antiderivative of MC to get C.

$$C = 100x + 2x^2 + D$$

for some constant D. Since $C(100) = 35000$, $D = 5000$. So, we have

$$C(x) = 100x + 2x^2 + 5000$$