Math 211

Review Problem 1

1. Find $\dfrac{dy}{dx}$. Do NOT SIMPLIFY your answer in this problem.

   (a) $y = (x^2 - x)(x^{\frac{2}{3}} + 2x + 5)$

   Solution

$$y' = (2x-1)(x^{\frac{2}{3}} + 2x + 5) + (x^2 - x)(\frac{2}{3}x^{-\frac{1}{3}} + 2)$$

   (b) $y = \dfrac{5x+4}{\sqrt{6x^3+7}}$

   Solution

$$\frac{5(\sqrt{6x^3+7}) - \frac{(5x+4)(6x^2)}{2\sqrt{6x^3+7}}}{6x^3+7}$$

   (c) $y = x^3\sqrt{x^2-5x-10}$

   Solution

$$3x^2\sqrt{x^2-5x-10} + \frac{x^3(2x-5)}{2\sqrt{x^2-5x-10}}$$

   (d) $y = \dfrac{(5x^3-6x^2)^4}{\sqrt{3x^2-8x-1}}$

   Solution

$$\frac{4(5x^3-6x^2)^3(15x^2-12x)\sqrt{3x^2-8x-1} - \frac{(5x^3-6x^2)^4(6x-8)}{2\sqrt{3x^2-8x-1}}}{3x^2-8x-1}$$

2. (a) $y = \sqrt{2x+1}$. Find $\dfrac{dy}{dx}$ by using the definition of the derivative.

   Solution

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} \\
&= \lim_{h\to 0}\frac{(\sqrt{2x+2h+1} - \sqrt{2x+1})(\sqrt{2x+2h+1} + \sqrt{2x+1}}{h(\sqrt{2x+2h+1} + \sqrt{2x+1}} \\
&= \lim_{h\to 0}\frac{2x+2h+1-(2x+1)}{h(\sqrt{2x+2h+1} + \sqrt{2x+1}} \\
&= \lim_{h\to 0}\frac{2h}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})} \\
&= \lim_{h\to 0}\frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} \\
&= \frac{1}{\sqrt{2x+1}}
\end{aligned}$$

   (b) $f(x) = \dfrac{2}{3x+5}$. Find $f'(10)$ by using the definition of the derivative.

   Solution

$$\begin{aligned}
f'(10) &= \lim_{h\to 0}\frac{f(10+h) - f(10)}{h} \\
&= \lim_{h\to 0}\frac{\frac{2}{35+h} - \frac{2}{35}}{h}
\end{aligned}$$

$$= \lim_{h\to 0} \frac{\frac{70-2(35+h)}{35(35+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{35(35+h)} \frac{1}{h}$$

$$= \lim_{h\to 0} -\frac{2}{35(35+h)}$$

$$= -\frac{2}{1225}$$

3. Find the indicated limit.

   (a) $\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x}$.
   Solution

$$\frac{\sqrt{x+9}-3}{x} = \frac{(\sqrt{x+9}-3)(\sqrt{x+9}+3)}{x(\sqrt{x+9}+3)} = \frac{1}{\sqrt{x+9}+3}$$

   So,

$$\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x+9}+3} = \frac{1}{6}$$

   (b) $\lim_{x\to\infty} \frac{3000x^5 - 8x^2 + 2x + 1}{5x^6 + 2x + 12} = \lim_{x\to\infty} \frac{\frac{3000}{x} - \frac{8}{x^4} + \frac{2}{x^5} + \frac{1}{x^6}}{5 + \frac{2}{x^5} + \frac{12}{x^6}} = 0$

   (c) $\lim_{x\to\infty} \frac{3x^8 + 5x^3 + 9}{1000x^6 - 35x^3 + 12} = \lim_{x\to\infty} \frac{3 + \frac{5}{x^5} + \frac{9}{x^8}}{\frac{1000}{x^2} - \frac{35}{x^5} + \frac{12}{x^8}} = \infty$

   (d) $\lim_{x\to 10} \frac{x^2-1}{x^2+1} = \frac{100^2-1}{100^2+1} = \frac{99}{101}$

   (e) $\lim_{x\to 1} \frac{\frac{1}{x}-1}{x-1} = \lim_{x\to 1} \frac{(\frac{1}{x}-1)x}{(x-1)x} = \lim_{x\to 1} \frac{1-x}{(x-1)x} = \lim_{x\to 1} \left(-\frac{1}{x}\right) = -1$

4. Find the value of a that makes the following function continuous whenever $0 \le x \le 20$?

$$f(x) = \begin{cases} 2x + a & (0 \le x \le 10) \\ 4x + 25 & (10 < x \le 20) \end{cases}$$

   SolutionWe need to have $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x)$. $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^-} (2x+a) = 20 + a$ and $\lim_{x\to 10^+} f(x) = \lim_{x\to 10^+} (4x+25) = 65$. So $20 + a = 65$ and $a = 45$.

5. (a) Find the equation of the line that is tangent to the curve $y = x^2 - 4x + 10$ at $x = 5$.
   SolutionWe first find the slope at $x = 5$. $y' = 2x - 4$ and the slope is $2(5) - 4 = 6$. At $x = 5$, $y = 15$. So, the equation is

$$y - 15 = 6(x-5) \qquad y = 6x - 15$$

   (b) For what value of x is the slope 10 times what it is at $x = 5$?
   SolutionWe want find x satifying

$$2x - 4 = 10(6)$$

   So, $x = 32$.

6. A toy rocket is launched from the roof of a building. The height of the rocket above the ground t seconds after launch (and until crashing into the ground) is $s = 88 + 80t - 8t^2$ feet. Answer the followings, including appropriate unit in your answer.

   (a) How tall is the building?
   SolutionWhen $t = 0$, s is same as height. $s(0) = 88$.

(b) How fast is the rocket traveling after 1 second?
Solution$v = s' = 80 - 16t$. So, $v(1) = 64$.

(c) When does the rocket reach its maximum height?
SolutionNeed to find when $v = 0$. $80 - 16t = 0$. $t = 5$.

(d) How high is the rocket then?
Solution$s(5) = 88 + 80(5) - 8(5)^2 = 288$.

(e) How long will it take for its speed at $t = 6$ to double?
$v(6) = 80 - 16(6) = -16$. We want $v = 2(-16) = -32$. $80 - 16t = -32$. $t = 7$.

(f) How fast will it be going when it hits the ground?
When $s = 0$. $88 + 80t - 8t^2 = 0$. $t = -1, 11$. So it hit the ground when $t = 11$. $v(11) = 80 - 16(11) = -96$.

7. Demographers predict that the population of a certain urban region will follow the model

$$P(t) = \frac{Kt}{100 + t^2}, \qquad t > 0$$

for the next 20 years,where K is a positive constant.

(a) For which values of t(= time) will $P(t)$ increase? Decrease?
Solution$P'(t) = \frac{K(100+t^2)-Kt(2t)}{(100+t^2)^2} = \frac{K(-t^2+100)}{(100+t^2)^2}$. For $P(t)$ to increase, $P'(t) > 0$ Since the denominator $(100 + t^2)^2$ is always positive, we need to have

$$-t^2 + 100 > 0$$

If we solve this, $-10 < t < 10$. But, $t > 0$. So, for $0 < t < 10$, $P(t)$ increases. Clearly, $P(t)$ decreases for $t > 10$.

(b) Find the maximum value of $P(t)$.
SolutionFrom (a), we see that $P(t)$ has its maximun at $t = 10$. So, the maximum value of $P(t)$ is

$$\frac{10K}{200} = \frac{K}{20}$$

8. The function $f(x) = 2x^3 - ax^2 + 6$ decreases **only** on the interval $(0, 3)$.

(a) Find the number a.
SolutionWe want the following situation:

x	$x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
$f'(x)$	+	0	−	0	+
$f(x)$	inc	max	dec	min	inc

$f'(x) = 6x^2 - 2ax$. We want $f'(0) = 0$ and $f'(3) = 6(3)^2 - 2a(3) = 0$. $a = 9$.

(b) Find its relative maximum value.

(c) Find its relative minimum value.
SolutionFrom above, max is at $x = 0$ and min is at $x = 3$. So, max $= f(0) = 6$ and min $= f(3) = 2(3)^3 - 9(3)^2 + 6 = -21$.

9. Describe the concavity of the graph of the following function f and find the inflection points. Use this information to sketch the graph of f, where possible.

$$f(x) = x^2 - x^3 + 3x - 6$$

Solution$f'(x) = 2x - 3x^2 + 3$ and $f''(x) = 2 - 6x$. To find the inflection points, $f''(x) = 2 - 6x = 0$. $x = 1/3$.

10. Find all asymptotes, including oblique asymptotes of the following function f.

$$f(x) = 6 - x + \frac{x+3}{1-x^2}$$

Solution

(a) (Horizontal) $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$. No horizontal asymptotes.

(b) (Vertical) $1 - x^2 = 0$. $x = 1$ and $x = -1$.

(c) (Oblique) Since $\lim_{x\to\infty} \frac{x+3}{1-x^2} = 0$ and $\lim_{x\to-\infty} \frac{x+3}{1-x^2} = 0$, $y = 6 - x$ is a oblique asymptote.